

BOOLEAN REPRESENTATIONS OF MATROIDS AND LATTICES

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ABSTRACT. We introduce a new representation concept for lattices by boolean matrices, and utilize it to prove that any matroid is boolean representable. We show that such a representation can be easily extracted from a representation of the associated lattice of flats of the matroid, leading also to a tighter bound on the representation's size. Consequently, we obtain a linkage of boolean representations with geometry in a very natural way.

INTRODUCTION

A matroid is a combinatorial structure that generalizes the familiar notion of independence in classical linear algebra; this structure arises often in many branches of pure and applied studies [14, 15, 23]. The classical theory of matroid representations essentially deals with the realization of matroids as “vector spaces”, allowing therefore the utilization both of algebraic and geometric tools in matroid theory; matroids that do have such a realization are termed representable. See [16, 23].

Over the years much effort in the study of this classical representation theory has been invested in the attempt to specify classes of matroids that are realizable as vector spaces defined over fields [25]. The understanding that not all matroids are representable in the field sense (see [15]) has provided the motivation for considering “vector spaces” built over other “weaker” ground structures instead of a field, for example a partial field [19] or a quasi field [6]. Yet, the representations taking place over these structures provide an incomplete result – they do not capture all matroids. As shown in [10], the superboolean semiring provides a complete appropriate alternative framework to that of fields which have customarily served for matroid representations [20, 24, 25].

Superboolean representations, and more generally supertropical representations, of hereditary collections have been studied initially in [10] where it was shown that every hereditary collection has a superboolean representation and hence every matroid does also. (Hereditary collections, also known as finite abstract simplicial complexes, are a much wider class of objects which includes matroids.)

The present paper stresses further the study of boolean representations, focusing on matroids and their associated lattices; these representations are much simpler and more accessible for computations. Our main result is:

Theorem 4.2. *Any matroid is boolean-representable.*

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The proof of the theorem makes crucial use of the fundamental connection, which is at the heart of our new approach, between lattices and boolean representations, leading to a new representation concept of finite lattices (Definitions 3.1 and 3.2). Incorporating this lattice representation perspective, applied to the geometric lattice of flats of a matroid, we obtain a boolean representation for any matroid (Theorem 4.1). In particular, we provide an explicit way to directly extract such matroid representations from boolean representations of the associated lattices of flats of matroids (Theorem 4.1).

The novel concept of lattice representations by boolean matrices is presented in §3.1, leading naturally to the new notions of c -independence and c -rank of lattices (Definition 3.2). These notions are also applicable to other abstract structures such as semilattices and partial ordered sets, as studied in detail in [9, 18].

We show that our new notion of c -independence for lattices, yielding also the notion of c -rank, is properly compatible with the length of chains of a lattice. As a consequence, the c -rank of the representation of a lattice equals the height of the lattice (Theorem 3.6). This result has a deeper meaning, it establishes a significant correspondence between c -dependence of sup-generating subsets of a lattice and its partitions (§3.2, also see Theorem 3.12). These correspondences provide a strong evidence that our new notions are the appropriate ones for the working with lattices, and has strong connections with chamber systems

The way of representing matroids via their flat-lattice establishes an easy systemic construction procedure. Moreover, it gives us a tighter upper bound on the representation size of matroids (cf. §4.2). Employing a natural embedding of the boolean semiring in the tropical semiring, our results are easily generalized further, showing that all matroids are tropically representable as well (Corollary 4.6). Furthermore, more generally, this result also holds for any idempotent semiring.

The extra benefit arises from our development is the important linkage between boolean matrices and geometry, established naturally by use of geometric lattices which are lattices of flats of matroids [16, Theorem 1.7.5].

Some indication on the connections of our results with previous research (up to the authors limited knowledge) is in order. The first author in [8] in 2006 defined the notion of independence for columns (rows) of a matrix with coefficients in a supertropical semiring. Restricting this notion to the superboolean semiring \mathbb{SB} and further to the subset of boolean matrices, we obtain the notion of independence for columns of a boolean matrix, to be used in this paper. Around mid 2008 the second author saw how to apply [8] in other areas of mathematics, which later led to this paper and [9, 10].

Much earlier, in the 1990's, unknown by the first author, Dress and Wenzel [5, 21] had also isolated the superboolean semiring as the correct semiring of coefficients for matroids, see [21, pp164]. They showed that every matroid $\mathcal{M} := (E, \mathcal{H})$ gives rise to a Grassmann-Plücker map from E to \mathbb{SB} . This “determinant-type” map can be used to define independent sets of a matroid, and in Whitney [24] or in §3.1 below, instead of the classical determinant. Thus, each matroid has its own determinant-type map.

Our approach is extremely different. Although, as Dress and Wenzel, we require the coefficients to be superboolean, we choose one determinant-type map for all matroids, that is the permanent of square superboolean matrices (cf. Lemma 1.3). Then we naturally adopt Whitney's [24] approach, only that he uses classical determinant while we use permanent. Note however that permanent is not a Grassmann-Plücker function. This is more in the flavor of [4] than Dress and Wenzel. Also the authors independently rediscovered related results (cf. §3.2) of Björner and Ziegler from 1991 [2] on taking transversals on the partitions defined by maximal chains in the lattice of flats.

The following comments on axioms and representations of hereditary collection and matroids may be helpful for the reader.

Axioms	Representations	Comments
Matroid (Whitney [24])	Matrices over fields	– Strong exchange axiom – Weak representations – Representation size equals the matroid rank – Not all matroids are representable
Hereditary collection (Def. 1.8)	Superboolean matrices (See [10])	– Weak axioms – Very strong representations – Every hereditary collection is representable – Applications of this result are needed
Point replacement (PR, Def. 1.10)	Boolean matrices	– All matroids are representable (strong) – Interesting to determine which hereditary collections have boolean representations to strengthen the PR axiom. – Boolean matrices representation are probably not that far from matroid.

Notation. In this paper, for simplicity, we use the following notation: Given a subset $X \subseteq E$, and elements $x \in X$ and $p \in E$, we write $X - x$ and $X + y$ for $X \setminus \{x\}$ and $X \cup \{y\}$, respectively; accordingly we write $X - x + y$ for $(X \setminus \{x\}) \cup \{y\}$.

1. PRELIMINARIES

1.1. Boolean and superboolean algebras. The **superboolean semiring** $\mathbb{SB} := (\{1, 0, 1^\nu\}, +, \cdot)$ is a three element supertropical semiring [11], a “cover” of the familiar boolean semiring $\mathbb{B} := (\{0, 1\}, +, \cdot)$, endowed with the two binary operations:

$+$	0	1	1^ν	\cdot	0	1	1^ν
0	0	1	1^ν	0	0	0	0
1	1	1^ν	1^ν	1	0	1	1^ν
1^ν	1^ν	1^ν	1^ν	1^ν	0	1^ν	1^ν

addition and multiplication, respectively. This semiring is totally ordered by $1^\nu > 1 > 0$. Note that the boolean semiring \mathbb{B} is an idempotent semiring, while \mathbb{SB} is not, since $1 + 1 = 1^\nu$; thus \mathbb{B} is **not** a subsemiring of \mathbb{SB} . The element 1^ν is called the **ghost** element of \mathbb{SB} , where $\mathcal{G}_0 := \{0, 1^\nu\}$ is the **ghost ideal** of \mathbb{SB} . (See [10] for more details.)

Superboolean matrices are matrices with entries in \mathbb{SB} , defined in the standard way, where addition and multiplication (respecting matrix sizes) are induced from the operations of \mathbb{SB} as in the familiar matrix construction. A typical matrix is often denoted as $A = (a_{i,j})$, and the zero matrix is written as (0) . A boolean matrix is a matrix with coefficients in $\{0, 1\}$. In what follows, these matrices are considered as superboolean matrices with entries in the subset $\{0, 1\} \subset \mathbb{SB}$. The reader should keep in mind that the boolean matrices are only a subset of the superboolean matrices and **not** a subsemiring.

Definition 1.1. The **complement** $A^c := (a_{i,j}^c)$ of a superboolean matrix $A = (a_{i,j})$ is defined by the rule:

$$a_{i,j}^c = 1 \iff a_{i,j} = 0.$$

The **transpose** $A^t = (a_{i,j}^t)$ of A is given by $a_{i,j}^t = a_{j,i}$.

Note that by this definition, we have $a_{i,j}^c = 1^\nu = a_{i,j}$.

Proposition 1.2. Transposition and complement commute, i.e., $(A^t)^c = (A^c)^t$ for any $n \times n$ superboolean matrix A .

Proof. Straightforward, $(a_{i,j}^t)^c = (a_{j,i})^c = (a_{i,j}^c)^t$. \square

We define the **permanent** of an $n \times n$ superboolean matrix $A = (a_{i,j})$ as in the standard way:

$$\text{per}(A) := \sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n}, \quad (1.1)$$

where S_n stands for the group of permutations of $\{1, \dots, n\}$. Accordingly, **the permanent of a boolean matrix can be 1^ν** . We say that a matrix A is **nonsingular** if $\text{per}(A) = 1$, otherwise A is said to be **singular** [7].

Lemma 1.3 ([10, Lemma 3.2]). *An $n \times n$ matrix is nonsingular iff by independently permuting rows and columns it has the triangular form*

$$A' := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ * & \cdots & * & 1 \end{pmatrix}, \quad (1.2)$$

with all diagonal entries 1, all entries above the diagonal are 0, and the entries below the diagonal belong to $\{1, 1^\nu, 0\}$.

Let A be an $m \times n$ superboolean matrix. We say that an $k \times \ell$ matrix B , with $k \leq m$ and $\ell \leq n$, is a **submatrix** of A if B can be obtained by deleting rows and columns of A . In particular, a **row** of a matrix A is an $1 \times n$ submatrix of A , where a **subrow** of A is an $1 \times \ell$ submatrix of A , with $\ell \leq n$.

The following definition is key to all that follows; it includes the definition of when a subset of columns (rows) of a superboolean matrix are independent.

Definition 1.4 ([8, Definition 1.2]). *A collection of vectors $v_1, \dots, v_m \in \mathbb{SB}^{(n)}$ is said to be (linearly) **dependent** if there exist $\alpha_1, \dots, \alpha_m \in \{0, 1\}$, not all of them 0, for which $\alpha_1 v_1 + \cdots + \alpha_m v_m \in \mathcal{G}_0^{(n)}$. Otherwise the vectors are said to be **independent**.*

The **column rank** of a superboolean matrix A is defined to be the maximal number of independent columns of A . The **row rank** is defined similarly with respect to the rows of A .

Theorem 1.5 ([8, Theorem 3.11]). *For any superboolean matrix A the row rank and the column rank are the same, and this rank is equal to the size of the maximal nonsingular submatrix of A .*

Corollary 1.6 ([10, Corollary 3.4]). *A subset of k columns (or rows) of A is independent iff it contains a $k \times k$ nonsingular submatrix.*

In the sequel, we use the following notations for submatrices:

Notation 1.7. *We write $A[*, Y]$ for the submatrix of A having the column subset $Y \subseteq \text{Col}(A)$, which sometimes is refer to as a collection of vectors, but no confusion should arise. Similarly, we write $A[X, *]$ for the submatrix of A having the row subset $X \subseteq \text{Row}(A)$, also refer to as a collection of vectors. We define $A[X, Y]$ to be the submatrix of A having the intersection of columns Y and the row subset $X \subseteq \text{Row}(A)$, often also referred to as a collection of sub-vectors.*

1.2. Hereditary collections. We write $|E|$ for the cardinality of a given finite ground set E , and $\text{Pw}(E)$ for the **power set** of E . In what follows, unless otherwise is specified, we always assume that $|E| = n$, and thus have $|\text{Pw}(E)| = 2^n$. Subsets of E of cardinality k are termed k -sets, for short.

Definition 1.8. *A **hereditary collection** (or a finite abstract simplicial complex) is a pair $\mathcal{H} := (E, \mathcal{H})$, with E finite and collection $\mathcal{H} \subseteq \text{Pw}(E)$, that satisfies the axioms:*

- HT1: \mathcal{H} is nonempty,
 HT2: $X \subseteq Y, Y \in \mathcal{H} \Rightarrow X \in \mathcal{H}$.

A subset $X \in \mathcal{H}$ is said to be **independent**, otherwise $X \notin \mathcal{H}$ is called **dependent**. A minimal dependent subset (with respect to inclusion) of E is called a **circuit**, the collection of all circuits of a hereditary collection \mathcal{H} is denoted by $\mathcal{C}(\mathcal{H})$. A maximal independent subset (with respect to inclusion) is called a **basis** of a hereditary collection \mathcal{H} . The set of all bases of \mathcal{H} is denoted as $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{H}$ and termed the **basis set** of \mathcal{H} . The **rank** $\text{rk}(\mathcal{H})$ of \mathcal{H} is defined to be the cardinality of the largest member of the basis set $\mathcal{B}(\mathcal{H})$ of \mathcal{H} .

Definition 1.9. Hereditary collections $\mathcal{H}_1 = (E_1, \mathcal{H}_1)$ and $\mathcal{H}_2 = (E_2, \mathcal{H}_2)$ are said to be **isomorphic** if there exists a bijective map $\varphi : E_1 \rightarrow E_2$ that respects independence; that is

$$\varphi(X_1) \in \mathcal{H}_2 \Leftrightarrow X_1 \in \mathcal{H}_1, \quad \text{for any } X_1 \subseteq E_1.$$

Given a hereditary collection, we recall the following axiom:

Definition 1.10. We say that a hereditary collection $\mathcal{H} = (E, \mathcal{H})$ satisfies the **point replacement property** iff

PR: For every $\{p\} \in \mathcal{H}$ and every nonempty subset $J \in \mathcal{H}$ there exists $x \in J$ such that $J - x + p \in \mathcal{H}$.

The existence of a boolean representation (to be defined next) of a hereditary collection implies the point replacement property.

Theorem 1.11 ([10, Theorem 5.3]). If a hereditary collection $\mathcal{H} = (E, \mathcal{H})$ has a \mathbb{B} -representation, then \mathcal{H} satisfies PR.

1.3. Representation of hereditary collection. Any $m \times n$ superboolean matrix A gives rise to a hereditary collection $\mathcal{H}(A)$ constructed in the following way: we label uniquely the columns of A (realized as vectors in $\mathbb{SB}^{(m)}$) by a set E , $|E| = n$, the independent subsets $\mathcal{H} := \mathcal{H}(A)$ of \mathcal{H} are then subsets of E corresponding to column subsets of A that are linearly independent in $\mathbb{SB}^{(m)}$, cf. Definition 1.4. Having Corollary 1.6, the independent subsets of $\mathcal{H}(A)$ can be described equivalently by using nonsingular submatrices, which we call **witnesses**:

WT: $Y \in \mathcal{H}(A) \Leftrightarrow \exists X \subseteq \text{Row}(A)$ with $|X| = |Y|$
such that $A[X, Y]$ is nonsingular.

We call $\mathcal{H}(A)$ an **SB-vector hereditary collection**, and say that it is a \mathbb{B} -vector hereditary collection when A is a boolean matrix, cf. [10, Definition 4.3].

A hereditary collection \mathcal{H}' is **superboolean-representable**, written **SB-representable**, if it is isomorphic (cf. Definition 1.9) to an SB-vector hereditary collection $\mathcal{H}(A)$ for some superboolean matrix A , and write $A(\mathcal{H})$ for an **SB-representation** of \mathcal{H} . When the matrix $A(\mathcal{H})$ is boolean, we call this representation a **boolean representation**, written **B-representation**, and say that \mathcal{H} is **B-representable**.

Theorem 1.12 ([10, Theorem 4.6]). Any hereditary collection is superboolean-representable.

Thus, the natural question becomes: which hereditary collections are boolean representable?

2. MATROIDS AND THEIR FLAT LATTICES

To make this paper reasonable self contained, we open with some classical definitions and results about matroids and their lattice of flats, see [14, 15, 16, 23].

Definition 2.1. A **matroid** $\mathcal{M} := (E, \mathcal{H})$ is hereditary collection that also satisfies the following exchange axiom:

MT: If X and Y are in \mathcal{H} and $|Y| = |X| + 1$, then there exists $y \in Y \setminus X$ such that $X + y$ is in \mathcal{H} .

A single element $x \in E$ that forms a circuit of $\mathcal{M} := (E, \mathcal{H})$, or equivalently it belongs to no basis, is called a **loop**. Two elements x and y of E are said to be **parallel**, written $x \parallel y$, if the 2-set $\{x, y\}$ is a circuit of \mathcal{M} . A matroid is called **simple** if it has no circuits consisting of 1 or 2 elements, i.e., has no loops and no parallel elements. This is equivalent to all subsets with 2 or less elements are independent.

The **closure** $\text{cl}(X)$ of a subset $X \subseteq E$ is the subset of E containing X and every element $y \in E \setminus X$ for which there is a circuit $C \subseteq X + y$ containing y . This defines a closure operator $\text{cl} : \text{Pw}(E) \rightarrow \text{Pw}(E)$ which has the **Mac Lane-Steinitz exchange property**:

For any $x, y \in E$ and all $Y \subseteq E$, if $x \in \text{cl}(Y + y) \setminus \text{cl}(Y)$, then $y \in \text{cl}(Y + x)$.

A subset $X \subseteq E$ is said to be **closed**, also called a **flat**, if $X = \text{cl}(X)$. The closed subsets of a matroid satisfy the following properties.

- (a) The whole ground set E is closed.
- (b) If X and Y are closed, then the intersection $X \cap Y$ is closed.
- (c) If X is a flat, then the flats Y that **cover** X , i.e., Y properly contains X without any flat Z between X and Y , partition the elements of $E \setminus X$.

The following proposition includes all the properties of flats we will need later in this paper.

Proposition 2.2. *For any matroid $\mathcal{M} := (E, \mathcal{H})$ we have:*

- (i) $\text{cl}(B) = E$ for any basis $B \in \mathcal{B}(\mathcal{M})$ of \mathcal{M} .
- (ii) If C is a circuit of a matroid, then, for all c in C , c is a member of $\text{cl}(C - c)$.
- (iii) X is independent in a matroid iff x is not a member of $\text{cl}(X - x)$ for all $x \in X$.
- (iv) Y is dependent in a matroid iff there exists an element $y \in Y$ such that y is a member of $\text{cl}(Y - y)$.
- (v) If $\mathcal{M} := (E, \mathcal{H})$ and $\mathcal{M}' := (E, \mathcal{H}')$ are matroids, and no circuit of \mathcal{M}' lies in \mathcal{H} , then and only then, \mathcal{H} is a subset of \mathcal{H}' .

Proof.

- (i): By maximality of independence, for each $x \in E \setminus B$, the subset $B + x$ has a circuit containing x .
- (ii): Clear from the definition of the closure $\text{cl}(X)$.
- (iii): X independent clearly implies that x is not a member of $\text{cl}(X - x)$, since X can not contain a circuit. Also X is dependent iff X contains a circuit C , so choosing c in C , implies that c is a member of $\text{cl}(X - c)$.
- (iv): The proof is logically equivalent to that of (iii).
- (v): The statement is logically equivalent to the definition of circuit.

□

The “smaller” flats of simple matroids are easily determined:

Remark 2.3. *When a matroid $\mathcal{M} := (E, \mathcal{H})$ is simple, the singleton $\{x\}$ is a flat for every $x \in E$, while \emptyset is the smallest flat (with respect to inclusion) of \mathcal{M} .*

The class of all flats of a simple matroid \mathcal{M} , partially ordered by set inclusion, forms a **matroid flat-lattice**, denoted as $\text{Lat}(\mathcal{M})$, having the **top element** $T = E$ and the **bottom element** $B = \emptyset$. The height $\text{ht}(\ell)$ of a lattice element $\ell \in \mathcal{L}$ is defined to be the length of the maximal chain from B to ℓ . A lattice element of height 1 counting edges, i.e., it covers the bottom element, is called an **atom**.

A finite lattice \mathcal{L} is **semimodular** if it satisfies the following conditions:

- (a) For every pair $\{\ell, m\}$ with $\ell < m$ all the chains from ℓ to m have the same length (called the **Jordan-Dedekind chain condition**);
- (b) $\text{ht}(\ell) + \text{ht}(m) \geq \text{ht}(\ell \vee m) + \text{ht}(\ell \wedge m)$, for any $\ell, m \in \mathcal{L}$.

A **geometric lattice** is a semimodular lattice in which every element is a join of atoms.

Lemma 2.4 ([16, Lemma 1.7.3]). *In a matroid flat-lattice $\text{Lat}(\mathcal{M})$, for all flats X, Y of \mathcal{M}*

$$X \wedge Y = X \cap Y \quad \text{and} \quad X \vee Y = \text{cl}(X \cup Y).$$

Theorem 2.5 ([16, Theorem 1.7.5]). *A lattice \mathcal{L} is geometric iff it is the lattice of flats of a matroid, i.e., a matroid flat-lattice.*

Corollary 2.6. *Every element of the lattice of flats of a matroid is join-generated by atoms.*

Remark 2.7. *In a matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, with $\mathcal{M} := (E, \mathcal{H})$ a simple matroid, every element $x \in E$ is closed and thus appears as an atom $\{x\}$ in \mathcal{L} . Thus, there is a one-to-one correspondence between the elements of E and the atoms of \mathcal{L} .*

3. REPRESENTATION OF LATTICES AND PARTITIONS

Unless otherwise is specified, in this paper we always assume **all lattices are finite** lattices, but almost all the results generalize easily to the infinite case.

3.1. Lattice representation. Within this part of the paper, when working with lattices, we realize a matrix as a semi-module (see [17, §8, §9], called there a boolean module), sup-generated by the matrix rows (or columns); therefore, as explained below, considering lattice representations we work row-wise.

Given a finite lattice $\mathcal{L} := (L, \leq)$, where $|L| = m$, we define the $m \times m$ boolean matrix $A_{\text{stc}}(\mathcal{L}) := (a_{i,j})$, which we called the **structure matrix of \mathcal{L}** , by the rule

$$a_{i,j} := \begin{cases} 1 & \text{if } \ell_i \leq \ell_j, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

Accordingly, such a structure matrix has the proprieties:

- (a) $a_{i,i} = 1$ for every $i = 1, \dots, n$, by reflexivity of \mathcal{L} ;
- (b) $a_{i,j} = 1$ iff $a_{j,i} = 0$ for any $i \neq j$, by antisymmetry of \mathcal{L} .

Clearly, using the setting (3.1), the structure of a lattice \mathcal{L} is uniquely recorded by the matrix $A_{\text{stc}}(\mathcal{L})$ and vice versa. Therefore, we identify the lattice \mathcal{L} with the structure matrix $A := A_{\text{stc}}(\mathcal{L})$. This leads us to the next two key definitions, playing a major role in our representation theory.

The reason that “c” occurs in the next definitions is basically because boolean modules are separative (the dual space of all sup-maps into the boolean semiring \mathbb{B} separates points) and the dual space is isomorphic to the original module with the order reversed, see [17, Chapter 9]. The same idea of passing to c is used in [9]

Also the matrix A_{stc} is triangular with ones on the diagonal, namely is nonsingular in our sense ($\text{per}(A_{\text{stc}}) = 1$) which in the field sense is invertible (i.e., $\det(A_{\text{stc}})$), the basis of Rota’s Mobius Inversion Theorem. Therefore, if A_{stc} was used instead of applying c , all subsets of lattice elements not containing the bottom would be independent, clearly the wrong choice.

(Key) Definition 3.1. *The **boolean representation** $A^c := A^c(\mathcal{L})$ of a finite lattice $\mathcal{L} := (L, \leq)$ is defined as*

$$A^c(\mathcal{L}) := (A_{\text{stc}}(\mathcal{L}))^c,$$

also written as $A^c := (a_{i,j}^c)$, cf. Definition 1.1.

This novel construction of boolean representation of lattices leads naturally to the following fundamental notions:

(Key) Definition 3.2. *The **c-rank** of a subset $W \subseteq L$ is then given by*

$$\text{c-rk}(W) := \text{rk}(A^c[W, *]), \quad A^c := A^c(\mathcal{L}),$$

where $A^c[W, *]$ stands for the rows of A^c corresponding to the subset W , cf. Notations 1.7. We say that a subset $W \subseteq L$ is **c-independent** if the rows $A^c[W, *]$ of the matrix A^c are independent in the sense of Definition 1.4, that is $\text{c-rk}(A^c[W, *]) = |W|$; otherwise we say that W is **c-dependent**.

It easy to verify that by this definition that c-rank is some sort of rank function, which always satisfies the relation

$$\text{c-rk}(W) \leq \text{c-rk}(\mathcal{L}) \leq |L|,$$

for every $W \subseteq L$. Actually, what the exact axioms are for this rank function is an important open research problem.

When a subset $W \subseteq L$ with $|W| = k$ is independent, the rows $A^c[W, *]$ of the representation $A^c := A^c(\mathcal{L})$ contain a $k \times k$ nonsingular submatrix $A^c[W, U]$ with $U \subseteq \mathcal{L}$, where $|U| = k$ (cf. Theorem 1.5), which we call a **witness** of W (in A^c). Abusing terminology, we also say that U is a witness of W in \mathcal{L} . Permuting independently the rows and columns of a witness, it has the triangular Form (1.2), cf. Lemma 1.3.

Note 3.3. *Although the work with matroids is performed column-wise, when considering lattices, in order to be compatible with the order, recorded by structure matrix, cf. (3.1), we have adopted a row-wise approach. As will be seen later, when working with the matroid flat-lattice, this approach fits well with the column-wise representations of matroid.*

Aiming to establish the correspondence between the height and the c-rank of lattices, we need the next lemmas.

Lemma 3.4. *Any strict chain $\ell_1 < \dots < \ell_k$, where $B < \ell_1$, of a lattice $\mathcal{L} := (L, \leq)$ determines a c-independent subset $W := \{\ell_1, \dots, \ell_k\}$ in \mathcal{L} .*

Proof. If $\ell_1 < \dots < \ell_k$ is a chain in \mathcal{L} , then ℓ_1, \dots, ℓ_k are independent with witness $U := \{m_1, \dots, m_k\}$, where $m_1 = B$, $m_2 = \ell_1$, \dots , $m_k = \ell_{k-1}$. \square

Lemma 3.5. *A witness of a c-independent subset $W := \{\ell_1, \dots, \ell_k\} \subseteq L$ of a lattice $\mathcal{L} := (L, \leq)$ gives rise to a strict chain of \mathcal{L} . Detail in the proof below.*

Proof. Let $A^c := A^c(\mathcal{L})$ be the matrix representation of \mathcal{L} , and suppose $A^c[W, U]$, where $U := \{m_1, \dots, m_k\}$, is a witness of W . Permuting independently rows and columns of A^c , we may assume that $A^c[W, U]$ is of the triangular form (1.2). Then, the chain

$$\tilde{m}_1 < \tilde{m}_2 < \dots < \tilde{m}_k, \quad \tilde{m}_j = m_j \wedge \dots \wedge m_k, \quad (3.2)$$

is a strict chain in \mathcal{L} . Indeed, since $\ell_1, \dots, \ell_{k-1} \leq m_k$, $\ell_k \not\leq m_k$, in particular $m_k = \tilde{m}_k$, and we inductively have:

$$\begin{array}{lll} \ell_1, \ell_2, \dots, \ell_{k-2} & \leq & \tilde{m}_{k-1}, & \ell_{k-1}, \ell_k & \not\leq & \tilde{m}_{k-1}, \\ \ell_1, \dots, \ell_{k-3} & \leq & \tilde{m}_{k-2}, & \ell_{k-2}, \ell_{k-1}, \ell_k & \not\leq & \tilde{m}_{k-2}, \\ \vdots & & \vdots & & & \vdots \\ \ell_1 & \leq & \tilde{m}_2, & \ell_2, \dots, \ell_{k-1}, \ell_k & \not\leq & \tilde{m}_2, \end{array} \quad (3.3)$$

and $\ell_1 \not\leq \tilde{m}_1 = m_1 \wedge \dots \wedge m_k$. \square

Theorem 3.6. $\text{c-rk}(\mathcal{L}) = \text{ht}(\mathcal{L})$ for any finite meet-closed lattice $\mathcal{L} := (L, \leq)$.

Proof. Apply Lemmas 3.4 and 3.5 respectively to a maximal strict chain and to a basis of \mathcal{L} . \square

Corollary 3.7. *Suppose $\mathcal{L} := \text{Lat}(\mathcal{M})$ is the matroid flat-lattice of $\mathcal{M} := (E, \mathcal{H})$, then $\text{rk}(\mathcal{M}) = \text{c-rk}(\mathcal{L}) = \text{ht}(\mathcal{L})$.*

In the present paper, to simplify the exposition, we have dealt mainly with matroid flat-lattices, i.e., geometric lattices, which are sufficient for the purpose of matroid representations. However, a similar idea of boolean lattice representations is applicable for much more general classes of lattice such as sup-generated lattices. In [9] we develop the theory of lattice representations in more generality, as well as representations of semilattices and partial ordered sets.

3.2. Matroid lattices and partitions. Björner and Ziegler in [2] have earlier results related to the results of this section which we obtained independently.

The notion of parallel elements of matroid introduces an equivalence relation on the ground set E , and thus on the matroid $\mathcal{M} := (E, \mathcal{H})$. Deleting all the loops of \mathcal{M} and then considering the equivalence classes $\tilde{E} := E/\parallel$ under the relation \parallel , we get a new matroid $\tilde{\mathcal{M}}$, cf. [16, §1.7]. Thus, by passing to the equivalent classes \tilde{E} , we may assume that \parallel is the identity, which implies that \mathcal{M} is simple. Having this perspective, in the sequel, we always assume that **all matroids are simple**.

Given a matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, with $\mathcal{M} := (E, \mathcal{H})$ a simple matroid, then \mathcal{L} is geometric and meet-closed, cf. Lemma 2.4. Recall that the elements of \mathcal{L} are flats of \mathcal{M} , and thus \mathcal{L} has the bottom element $B = \emptyset$ and the top element $T = E$; for notational convenience, we denote these flats of \mathcal{L} by F_i while the atoms of \mathcal{L} are sometimes denoted also as ℓ_j . Moreover, this lattice is join-generated by the set of atoms

$$\hat{E} := \text{Atom}(\mathcal{M}) = \{\{x_1\}, \dots, \{x_n\}\}, \quad x_i \in E.$$

For a matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, an edge (F_i, F_{i-1}) of \mathcal{L} corresponds to pair of flats of \mathcal{M} , where F_i covers the flat F_{i-1} . We assign to each edge (F_i, F_{i-1}) of \mathcal{L} the set theoretic difference

$$Q_i := F_i \setminus F_{i-1}. \quad (3.4)$$

Then, given a *maximal* (strict) chain

$$E = F_k > F_{k-1} > \dots > F_1 > F_0 = \emptyset, \quad k := \text{ht}(\mathcal{L}), \quad (3.5)$$

of \mathcal{L} , from top to bottom in \mathcal{L} , it is easy to see that these subsets Q_i are disjoint and their union equals E .

We call the collections

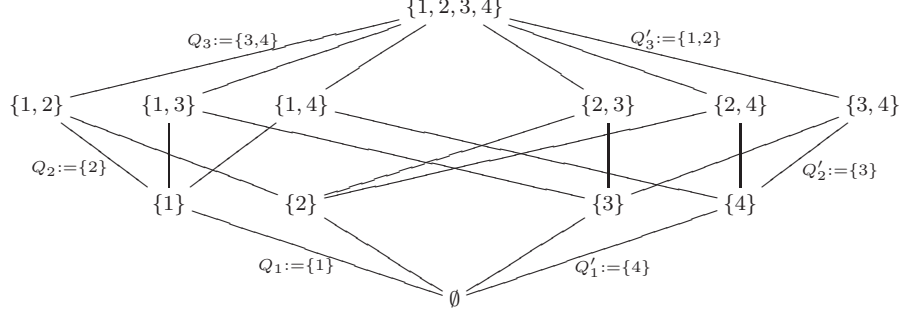
$$\mathcal{Q} := Q_1, \dots, Q_k, \quad k = \text{ht}(\mathcal{L}),$$

the **partitions** of E . Note that since $\mathcal{L} := \text{Lat}(\mathcal{M})$ is semimodular, all the partitions of E are of the same size, equals the height of \mathcal{L} . Abusing notation we also say that \mathcal{Q} is a partition of the matroid lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, with $\mathcal{M} := (E, \mathcal{H})$ a simple matroid.

Definition 3.8. *A subset $W = \{x_1, \dots, x_i\} \subseteq E$ is a **partial transversal** of a partition \mathcal{Q} iff each $x_j \in W$ lies in a distinct Q_i , i.e., $|W \cap Q_i| \leq 1$ for each $i = 1, \dots, k$. (In such a case, we also say that W is an **independent** set of the partition \mathcal{Q} .) A **basis** of a partition \mathcal{Q} is a partial transversal of maximal cardinality, equals the height of \mathcal{L} .*

A partial transversal may have less elements than the size of the partition. One easily sees that, by the pigeonhole principle, when a subset has a cardinality greater than the partition size (equals the number of blocks), then it can not be a partial transversal.

Example 3.9. Let $\mathcal{M} := U_{3,4}$ be the uniform matroid over 4 points, then the matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$ of \mathcal{M} is given by the diagram:



over 12 vertices, each corresponds to a flat of \mathcal{M} , which has 12 partitions.

Two partitions of E , $\mathcal{Q} = \{1\}, \{2\}, \{3,4\}$ and $\mathcal{Q}' = \{1,2\}, \{3\}, \{4\}$, are indicated on the corresponding edges of the diagram. The maximal partial transversals of the partition \mathcal{Q} , i.e., the bases, are $\{1,2,3\}$ and $\{1,2,4\}$. It easy to see that all the bases of the partitions are of cardinality 3.

The representation of this matroid flat-lattice is obtained by a 12×12 boolean matrix.

Remark 3.10. Let \mathcal{Q} be a partition of $\mathcal{L} := \text{Lat}(\mathcal{M})$.

- (i) If $X \subseteq E$ is a partial transversal of \mathcal{Q} , any subset $Y \subseteq X$ is also a partial transversal.
- (ii) When $X \subseteq E$ is a not partial transversal of \mathcal{Q} , any subset $Z \subseteq E$ containing X is not a partial transversal as well.

Lemma 3.11. If $W = \{x_1, \dots, x_t\} \subseteq E$ is a partial transversal of a partition $\mathcal{Q} := Q_1, \dots, Q_k$, with $Q_i := F_i \setminus F_{i-1}$, then the corresponding atom subset $\widehat{W} = \{\{x_1\}, \dots, \{x_t\}\} \subseteq \text{Atom}(\mathcal{L})$ is c -independent in \mathcal{L} with witness $U \subseteq \{F_0, \dots, F_{k-1}\}$.

Proof. It is enough to prove the lemma for W a basis of the partition \mathcal{Q} , i.e., $t = k$ having a witness $U = \{F_0, \dots, F_{k-1}\}$. Relabeling the elements of W , we may assume that $x_i \in Q_i$, $i = 1, \dots, k$. Let $\ell_i := \{x_i\}$ – the atoms of \mathcal{L} . Then, by construction, we have

$$\begin{array}{ll} \ell_1 & \leq F_1, F_2, \dots, F_{k-1}, & \ell_1 & \not\leq F_0, \\ \ell_2 & \leq F_2, \dots, F_{k-1}, & \ell_2 & \not\leq F_0, F_1, \\ \vdots & \vdots & \vdots & \vdots \\ \ell_{k-1} & \leq F_{k-1}, & \ell_{k-1} & \not\leq F_0, F_1, \dots, F_{k-2}, \end{array}$$

and $\ell_k \not\leq F_0, F_1, \dots, F_{k-1}$. Writing the matrix of these relations shows that U is a witness of W . \square

Theorem 3.12. A subset $W = \{x_1, \dots, x_t\} \subseteq E$ is a partial transversal of some partition $\mathcal{Q} := Q_1, \dots, Q_k$ iff $\widehat{W} = \{\{x_1\}, \dots, \{x_t\}\} \subseteq \text{Atom}(\mathcal{L})$ is c -independent in the matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, $\mathcal{M} := (E, \mathcal{H})$.

Proof. (\Rightarrow) : Immediate By Lemma 3.11.

(\Leftarrow) : Suppose W is not a partial transversal, and let $\overline{W} := \text{cl}(W)$ be the closure of W – a flat of \mathcal{M} . Thus, \overline{W} is a proper element of the flat-lattice \mathcal{L} , join-generated by a subset $\widehat{V} \subseteq \text{Atom}(\mathcal{L})$ of atoms of \mathcal{L} . Let \mathcal{L}' be the sublattice of \mathcal{L} consisting of all elements of \mathcal{L} below \overline{W} , and let $\mathcal{L}'|_{\widehat{W}}$ be the restriction of \mathcal{L}' to the join-generating subset $\widehat{W} \subseteq \widehat{V}$. Then, $\text{ht}(\mathcal{L}'|_{\widehat{W}}) < |\widehat{W}| = |W|$, since W is not a partial transversal. Thus, by Theorem 3.6, $c\text{-rk}(\widehat{W}) < |\widehat{W}|$, which means that \widehat{W} is dependent in \mathcal{L} . \square

Corollary 3.13. Independence of partial transversals and the c -independence of lattice coincide.

Corollary 3.14. *Maximal c -independent subsets of a lattice \mathcal{L} correspond to the bases of its partitions.*

4. REPRESENTATIONS OF MATROIDS

Having the method for boolean representation of lattices at hand, together with their connection to matroids, we can state our main result:

Theorem 4.1. *Given a simple matroid $\mathcal{M} := (E, \mathcal{H})$, let $A^c := A^c(\mathcal{L})$ be the boolean representation of the matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, and let $A^c|_{\text{Atom}(\mathcal{L})}$ be the restriction of A^c to the rows corresponding to the atoms of \mathcal{L} . Then, $(A^c|_{\text{Atom}(\mathcal{L})})^\dagger$ is a boolean representation of \mathcal{M} .*

Proof. Let $B := \{b_1, \dots, b_k\}$ be a basis of \mathcal{M} and consider the nested sequence of flats

$$\begin{aligned} F_k &:= \text{cl}(B), \\ F_{k-1} &:= \text{cl}(B \setminus \{b_k\}), \\ &\vdots \\ F_{k-j} &:= \text{cl}(B \setminus \{b_{k-j+1}, \dots, b_k\}), \\ &\vdots \\ F_1 &:= \text{cl}(B \setminus \{b_2, \dots, b_{k-1}\}), \\ F_0 &:= \emptyset. \end{aligned}$$

This flat sequence introduces a chain

$$\emptyset = F_0 < F_1 < \dots < F_{k-1} < F_k = E \quad (4.1)$$

in \mathcal{L} , where by Proposition 2.2.(iii), $\text{ht}(F_i) = i$ for each $i = 0, \dots, k$. (Note that we also have $k = \text{ht}(\mathcal{L})$.) Then, by construction, for each $i = 1, \dots, k$

$$b_i \in Q_i := F_i \setminus F_{i-1},$$

since adding b_i to F_{i-1} would increase the rank of F_{i-1} – contradicting the height of \mathcal{L} , cf. Theorem 3.6. Thus, the chain (4.1) is a maximal strict chain of \mathcal{L} . Moreover, it is also a witness for the independence of $\widehat{B} := \{\{b_1\}, \dots, \{b_k\}\}$, cf. Lemma 3.5. By the partition method (cf. §3.2), B is a partial transversal of the partition $\mathcal{Q} := Q_1, \dots, Q_k$ (cf. Definition 3.8) and is also a basis of \mathcal{Q} .

Since each independent subset $X \subset E$ of \mathcal{M} is contained in some basis B , the above argument shows that X is also independent in some partition of \mathcal{L} , and thus \widehat{X} is c -independent in the lattice representation $A^c(\mathcal{L})$ and in particular in the restriction $A^c|_{\text{Atom}(\mathcal{L})}$ to the rows corresponding to atoms of \mathcal{L} .

Let $C = \{x_1, \dots, x_t\} \subset E$ be a circuit, then by Proposition 2.2.(ii), x is in $\text{cl}(C - x_i)$ for every $x_i \in X$. We also know that $X \subseteq Y$ implies $\text{cl}(X) \subseteq \text{cl}(Y)$, since cl is a closure operator. Assume that X is a partial transversal of a partition $\mathcal{Q} := Q_1, \dots, Q_k$, where $k \geq t$, and suppose that $x_t \in Q_k = E \setminus F_{k-1}$, i.e., x_t is the closest element to the top in \mathcal{Q} up to reordering. Then, there is a flat Y containing $X - x_k$ but not x_k , and thus $\text{cl}(X - x_t) \subseteq \text{cl}(Y)$. But this is a contradiction since $x_t \in \text{cl}(X - x_t)$, which is a subset of Y . Then, $\widehat{X} := \{\{x_1\}, \dots, \{x_t\}\}$ is c -dependent in \mathcal{L} , by Theorem 3.12. Thus, we are done by Proposition 2.2.(v). \square

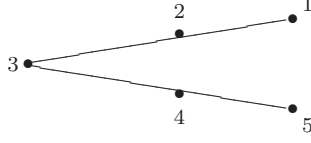
Composing the lattice representations of §3.1 with the extraction of matroid representations as in Theorem 4.1 we get the following:

Theorem 4.2. *Any matroid has a boolean representation.*

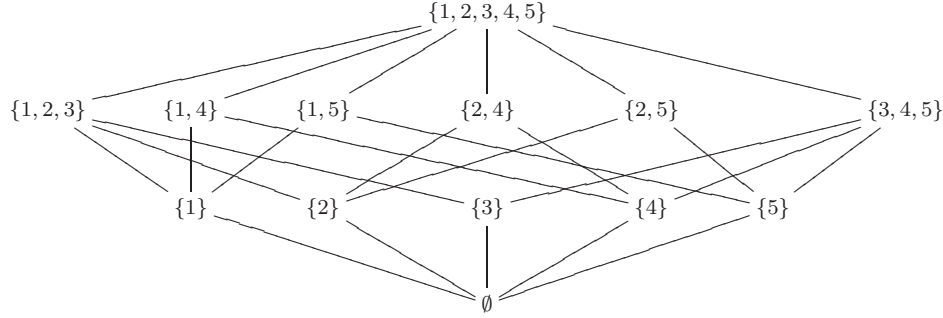
Proof. Any lattice $\mathcal{L} := (L, \leq)$, and in particular every matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$, has a boolean representation. The proof is then completed by Theorem 4.1. \square

4.1. Examples. We give some simple demonstrating examples of matroid representations, extracted from their lattice representations.

Example 4.3. Let \mathcal{M} be the simple matroid over the 5 point set $E := \{1, 2, 3, 4, 5\}$ whose bases are all the 3-subset except $\{1, 2, 3\}$ and $\{3, 4, 5\}$:



The matroid flat-lattice of $\mathcal{L} := \text{Lat}(\mathcal{M})$ associated to \mathcal{M} is then given by the following diagram



whose 13 flats are as listed above and the atoms are the singeltons subsets.

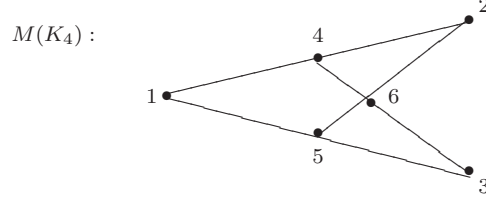
The representation $A^c := A^c(\mathcal{L})$ of the matroid flat-lattice $\mathcal{L} := \text{Lat}(\mathcal{M})$ is given by the following 13×13 boolean matrix:

\mathcal{L}	\emptyset	F_1	F_2	F_3	F_4	F_5	F_{14}	F_{15}	F_{24}	F_{25}	F_{123}	F_{345}	E
\emptyset	0	0	0	0	0	0	0	0	0	0	0	0	0
F_1	1	0	1	1	1	1	0	0	1	1	0	1	0
F_2	1	1	0	1	1	1	1	1	0	0	0	1	0
F_3	1	1	1	0	1	1	1	1	1	1	0	0	0
F_4	1	1	1	1	0	1	0	1	0	1	1	0	0
F_5	1	1	1	1	1	0	1	0	1	0	1	0	0
F_{14}	1	1	1	1	1	1	0	1	1	1	1	1	0
F_{15}	1	1	1	1	1	1	1	0	1	1	1	1	0
F_{24}	1	1	1	1	1	1	1	1	0	1	1	1	0
F_{25}	1	1	1	1	1	1	1	1	1	0	1	1	0
F_{123}	1	1	1	1	1	1	1	1	1	1	0	1	0
F_{345}	1	1	1	1	1	1	1	1	1	1	1	0	0
E	1	1	1	1	1	1	1	1	1	1	1	1	0

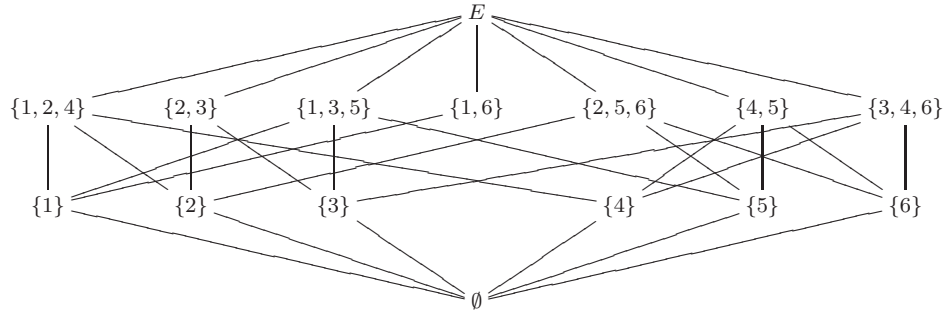
Taking the transpose of the restriction $B := A^c|_{\text{Atom}(\mathcal{L})}$ of A^c to the rows corresponding to the atoms of \mathcal{L} , and then omitting the rows of B^t corresponding to the atoms of \mathcal{L} we get a reduced representation of \mathcal{M} by the boolean matrix:

	F_1	F_2	F_3	F_4	F_5
\emptyset	1	1	1	1	1
F_{14}	0	1	1	0	1
F_{15}	0	1	1	1	0
F_{24}	1	0	1	0	1
F_{25}	1	0	1	1	0
F_{123}	0	0	0	1	1
F_{345}	1	1	0	0	0

Example 4.4. The matroid K_4 over the 6 point set $E := \{1, 2, 3, 4, 5, 6\}$, whose bases are all the 3-subset except $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 6\}$, and $\{2, 5, 6\}$ corresponds to the diagram:



The matroid flat-lattice $\mathcal{L} := \text{Lat}(K_4)$ of K_4 is given by the following diagram:



The representation $A^c := A^c(K_4)$ of \mathcal{L} is obtained by the following 15×15 boolean matrix:

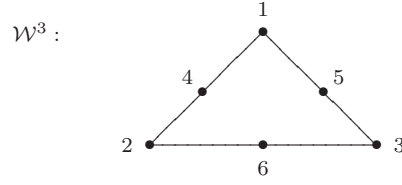
\leq	\emptyset	F_1	F_2	F_3	F_4	F_5	F_6	F_{16}	F_{23}	F_{45}	F_{124}	F_{135}	F_{256}	F_{346}	E
\emptyset	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_1	1	0	1	1	1	1	1	0	1	1	0	0	1	1	0
F_2	1	1	0	1	1	1	1	1	0	1	0	1	0	1	0
F_3	1	1	1	0	1	1	1	1	0	1	1	0	1	0	0
F_4	1	1	1	1	0	1	1	1	1	0	0	1	1	0	0
F_5	1	1	1	1	1	0	1	1	1	0	1	0	0	1	0
F_6	1	1	1	1	1	1	0	0	1	1	1	1	0	0	0
F_{16}	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0
F_{23}	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0
F_{45}	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0
F_{124}	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0
F_{135}	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0
F_{256}	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0
F_{346}	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

Taking the transpose of the restriction $B := A^c|_{\text{Atom}(\mathcal{L})}$ of A^c to the rows corresponding to the atoms of \mathcal{L} , and leaving the rows of B^t corresponding flats of cardinality ≥ 2 , we get the following

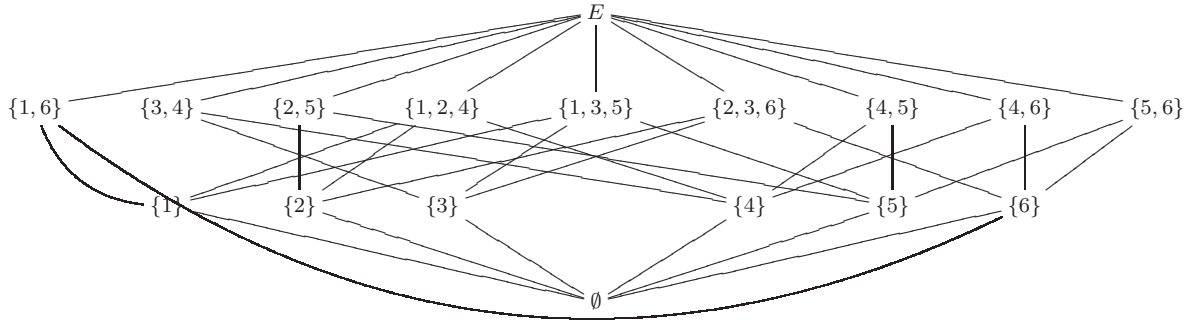
boolean representation of K_4 :

	F_1	F_2	F_3	F_4	F_5	F_6
\emptyset	1	1	1	1	1	1
F_{16}	0	1	1	1	1	0
F_{23}	1	0	0	1	1	1
F_{45}	1	1	1	0	0	1
F_{124}	0	0	1	0	1	1
F_{135}	1	1	0	1	0	1
F_{256}	1	0	1	1	0	0
F_{346}	1	1	0	0	1	0

Example 4.5. The matroid \mathcal{W}^3 over the 6 point set $E := \{1, 2, 3, 4, 5, 6\}$, whose bases are all the 3-subset except $\{1, 2, 4\}$, $\{1, 3, 5\}$, and $\{2, 3, 6\}$, has the diagram:



The matroid flat-lattice of $\mathcal{L} := \text{Lat}(\mathcal{W}^3)$ is as follows



The following 17×17 boolean matrix provides the representation $A^c := A^c(\mathcal{W}^3)$ of $\mathcal{L} := \text{Lat}(\mathcal{W}^3)$:

\leq	\emptyset	F_1	F_2	F_3	F_4	F_5	F_6	F_{16}	F_{25}	F_{34}	F_{45}	F_{46}	F_{56}	F_{124}	F_{135}	F_{236}	E
\emptyset	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_1	1	0	1	1	1	1	1	0	1	1	1	1	1	0	0	1	0
F_2	1	1	0	1	1	1	1	1	0	1	1	1	1	0	1	0	0
F_3	1	1	1	0	1	1	1	1	1	0	1	1	1	1	0	0	0
F_4	1	1	1	1	0	1	1	1	1	0	0	0	1	0	1	1	0
F_5	1	1	1	1	1	0	1	1	0	1	0	1	0	1	0	1	0
F_6	1	1	1	1	1	1	0	0	1	1	1	0	0	1	1	0	0
F_{16}	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0
F_{25}	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0
F_{34}	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0
F_{45}	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0
F_{46}	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0
F_{56}	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0
F_{124}	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0
F_{135}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0
F_{236}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
E	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

The transpose of the restriction $B := A^c|_{\text{Atom}(\mathcal{L})}$ of A^c to the rows corresponding to the atoms of \mathcal{L} gives us the following boolean representation of \mathcal{W}^3 :

$$A(\mathcal{W}^3) = \begin{array}{c|cccccc} & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \\ \hline \emptyset & 1 & 1 & 1 & 1 & 1 & 1 \\ F_{16} & 0 & 1 & 1 & 1 & 1 & 0 \\ F_{25} & 1 & 0 & 1 & 1 & 0 & 1 \\ F_{34} & 1 & 1 & 0 & 0 & 1 & 1 \\ F_{45} & 1 & 1 & 1 & 0 & 0 & 1 \\ F_{46} & 1 & 1 & 1 & 0 & 1 & 0 \\ F_{56} & 1 & 1 & 1 & 1 & 0 & 0 \\ F_{124} & 0 & 0 & 1 & 0 & 1 & 1 \\ F_{135} & 0 & 1 & 0 & 1 & 0 & 1 \\ F_{236} & 1 & 0 & 0 & 1 & 1 & 0 \end{array}$$

(Note that this representation can be reduced further by omitting duplicate rows.)

4.2. An upper bound on the representation size. Using the new representation of matroids, assisted by Corollary 3.7, we compute an upper bound for the size of the boolean matroids, i.e., the height of the representing boolean matrix. Let $A(\mathcal{M}) := A^c|_{\text{Atom}(\mathcal{L})}$, $\mathcal{L} := \text{Lat}(\mathcal{M})$, be a boolean representation of the matroid $\mathcal{M} := (E, \mathcal{H})$, as obtained from Theorem 4.1. Suppose \mathcal{M} has rank k and $A(\mathcal{M})$ is an $m \times n$ matrix, i.e., $|E| = n$. Then, we have the following naive upper bound:

$$m \leq \sum_{i=0}^k \binom{n}{i}.$$

Of course, a better upper bound on the size is the number of sji (strict join irreducibles, see [17, §6] and [9], also see [9]).

4.3. Tropical representations. In [10, Appendix A] we have shown that the boolean semiring $\mathbb{B} := (\{0, 1\}, +, \cdot)$ embeds naturally in the tropical semiring $\mathbb{R}_{(\max, +)} := (\mathbb{R} \cup \{-\infty\}, \max, +)$, or dually in $\mathbb{R}_{(\min, +)} := (\mathbb{R} \cup \{\infty\}, \min, +)$, and much more generally it embeds in any idempotent

semiring $S := (S, +, \cdot)$ by sending $1 \mapsto 1_S$ and $0 \mapsto 0_S$, the multiplicative unit and the zero of S , respectively. In particular, for the tropical semiring $S = \mathbb{R}_{(\max, +)}$, the embedding $\varphi : \mathbb{B} \hookrightarrow \mathbb{R}_{(\max, +)}$ is given by $\varphi : 1 \mapsto 0$ and $\varphi : 0 \mapsto -\infty$.

Having this embedding $\varphi : \mathbb{B} \hookrightarrow S$, we easily generalize the result of Theorem 4.2:

Corollary 4.6. *Every matroid is representable over any idempotent semiring, and in particular each matroid is tropically representable.*

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